

Henneberg constructions and covers of cone-Laman graphs

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Abstract

We give Henneberg-type constructions for three families of sparse colored graphs arising in the rigidity theory of periodic and forced symmetric frameworks. The proof method, which works with Laman-sparse finite covers of colored graphs highlights the connection between these sparse colored families and the well-studied matroidal (k, ℓ) -sparse families.

1. Introduction

Let $G = (V, E)$ be a finite directed graph, let Γ be a group, and let $\gamma = (\gamma_{ij})$ be an assignment of a “color” $\gamma_{ij} \in \Gamma$. The tuple (G, γ) is called a *colored graph*¹. In this paper, Γ will always be one of the abelian groups: $\mathbb{Z}/p\mathbb{Z}$, \mathbb{Z} , $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$, or \mathbb{Z}^2 . For these Γ , there is a well-defined homomorphism ρ from the cycle space $H_1(G, \mathbb{Z})$ to Γ , which we describe in Section 2. The number of linearly independent elements in the image of ρ on a subgraph G' of G is an invariant of G' which we define to be the ρ -rank of a subgraph. In this note, we study colored graphs defined by a hereditary sparsity property that depends on the ρ -rank. These generalize the well-studied (k, ℓ) -sparse graphs [5, 11], which are defined by the condition “ $m' \leq kn' - \ell$ ” for all subgraphs.

1.1. Sparse colored graphs Let (G, γ) be a colored graph, with n vertices and m edges. Further, let G' be an edge-induced subgraph with n' vertices, m' edges, ρ -rank r , and c'_i connected components with ρ -rank i (i will always be in $\{0, 1, 2\}$). Then (G, γ) is defined to be *Ross-sparse*² if, for all edge-induced subgraphs

$$m' \leq 2n' - 3c'_0 - 2(c'_1 + c'_2) \quad (1)$$

it is *cone-Laman-sparse* if, for all subgraphs

$$m' \leq 2n' - 3c'_0 - c'_1 - c'_2 \quad (2)$$

and it is *cylinder-Laman-sparse* if, for all subgraphs

$$m' \leq 2n' + r - 3c'_0 - 2(c'_1 + c'_2) \quad (3)$$

In, in addition (1) (resp. (2), (3)) hold with equality on the whole graph, then (G, γ) is a *Ross-graph*, (resp. *cone-Laman graph*, *cylinder-Laman graph*).

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¹Colored graphs are also known as “gain graphs” [23].

²An equivalent definition is due to Elissa Ross.

1.2. Inductive characterization These families can be characterized as the graphs generated from a fixed base by a sequence of several inductive moves. The moves we use are defined in Section 4 and illustrated in Figure 1.

Theorem 1. *A colored graph (G, γ) with:*

- \mathbb{Z}^2 colors is a Ross-graph if and only if it can be constructed from a base graph as in Figure 2(c) using the moves **(H1c)** and **(H2c)** [19].
- $\mathbb{Z}/p\mathbb{Z}$ colors, with p an odd prime, is a cone-Laman graph if and only if it can be constructed from a base graph as in Figure 2(a) using the moves **(H1c)**, **(H1c')**, and **(H2c)**.
- \mathbb{Z} colors is a cylinder-Laman-graph if and only if it can be constructed from a base graph as in Figure 2(b) using the moves **(H1c)** and **(H2c)**.

Results like this are known as *Henneberg constructions*, since they generalize a classical technique from [9] to all matroidal (k, ℓ) -sparse graphs [5, 11].

1.3. Interpretation of the colored Henneberg 2 move The somewhat technical nature of the colored-Henneberg move **(H2c)** has a more natural interpretation. Immerse the colored graph (G, γ) in \mathbb{R}^2/Γ with geodesic edges selected by the colors. The colored Henneberg move **(H2c)** then corresponds to putting the new vertex n on the edge ij that is being split and connecting n to its other neighbor k using the geodesic specified by the color on the new edge ik . This is a stronger statement than simply saying that there is *some* choice of coloring for the new edges would preserve the desired sparsity property.

1.4. Combinatorial rigidity motivation All the families of colored graphs described above arise from instances of the following geometric problem. A Γ -*framework* is a planar structure made of fixed-length bars connected by joints with full rotational freedom; additionally, it is symmetric under a representation of Γ by Euclidean motions of the plane, which induces a free Γ -action by automorphisms on the graph \tilde{G} that has as its edges the bars. The allowed motions preserve the length and connectivity of the bars *and* symmetry, but not necessarily the representation of Γ . A Γ -*framework* is *rigid* when the allowed motions are all Euclidean isometries and otherwise *flexible*. Generically, rigidity and flexibility are properties of the colored graph G that encodes \tilde{G} , and the “Maxwell-Laman question” (cf. [10, 16]) is to characterize the combinatorial types of generic, minimally rigid frameworks.

Justin Malestein and the author solved this problem for: *periodic frameworks* [13], where Γ is \mathbb{Z}^2 acting by translations with “flexible representation” of the translation lattice; for *crystallographic frameworks*, where Γ is generated by translations and a rotation of order 2, 3, 4 or 6 [14]. For the periodic case, the minimally rigid *colored-Laman graphs* are defined, using the notation above, by the counts

$$m' \leq 2n' + \max\{2r - 1, 0\} - 3c_0 - 2(c_1 + c_2) \quad (4)$$

At the time (4) had not been conjectured, nor, to the best of our knowledge, had matroidal families defined by counts of this form appeared in the combinatorial literature. The geometric idea leading to (4) is that a sub-framework that “sees” r flexible periods has $2r - 1$ non-trivial degrees of freedom from the lattice representation, $2n'$ from the coordinates of the vertices, and

each connected component has either two or three “trivial” motions commuting with any fixed lattice representation.³

The colored graph families under consideration here also correspond to generic minimally rigid frameworks in different forced-symmetric models: Ross graphs for *fixed-lattice periodic frameworks* [13, 19], which are periodic frameworks where the translation lattice is fixed; cone-Laman graphs for *cone frameworks* [14], where the symmetry group is a finite-order rotation around the origin; and cylinder-Laman graphs for *cylinder frameworks* [13, 15] which are periodic with one flexible period.

1.5. Novelty The combinatorial steps in [13, 14] rely on the “edge-doubling trick” of Lovász [12] and Recski [18] and then decompositions obtained by submodular function theory [4]. Seeing as the colored graph families under consideration arise in a planar rigidity setting, it is natural to ask what the connection they have to the well-studied $(2, 3)$ -sparse graphs (shortly *Laman graphs*) characterizing the minimally rigid planar frameworks [10].

It is not hard to see that the ρ -rank zero subgraphs must be $(2, 3)$ -sparse. On the other hand, the proof method employed here is based around the following proposition that characterizes a cone-Laman graph in terms of its symmetric cover:

Proposition 1.1. *Let p be an odd prime, and let (G, γ) be a $\mathbb{Z}/p\mathbb{Z}$ -colored graph with n vertices and $2n - 1$ edges. Then (G, γ) is a cone-Laman graph if and only if its symmetric lift (\tilde{G}, φ) is Laman-sparse.*

This connection between colored sparsity and (k, ℓ) -sparse covers is, to me, as interesting as Theorem 1. As an algorithmic consequence, if p is small relative to the number of vertices n , the algorithmic rigidity questions, as defined in [11], for cone-Laman graphs can all be solved in $O(n^2)$ time with the *pebble game* [2, 11]. It was in this context that the specialization of Proposition 1.1 $p = 3$ was first observed [1].

Proposition 1.1 doesn’t extend, naively⁴, at least, to the colored-Laman graphs of [13] or the Γ -colored-Laman graphs of [14]. Thus, we also obtain a distinction between the cone-Laman-sparse colored graph families and the more general ones introduced to understand periodic and crystallographic frameworks. Finding the “right” generalization of Proposition 1.1 would be very interesting.

1.6. Roadmap to the proof of Theorem 1 Proposition 1.1 allows us to study the combinatorial structure of cone-Laman graphs via the symmetric lift \tilde{G} . This allows us to apply the entire theory of Laman-sparse graphs apply. Colored Henneberg moves on a cone-Laman graph G correspond to “symmetrized groups” of uncolored Henneberg moves on \tilde{G} . The idea of symmetrizing Henneberg moves is not new [20], but the approach taken here is. The difficult step (Lemma 5.8) is to show that, after removing an entire vertex *orbit* in \tilde{G} , an entire edge *orbit* may be added to the remaining graph while maintaining Laman-sparsity. The proof makes use of a new circuit-elimination argument (Proposition 3.1) that avoids a complicated cases analysis.

1.7. Notations When dealing with the families of (uncolored, finite) (k, ℓ) -sparse graphs [11], we adopt the following conventions: (k, ℓ) -*circuits* are minimal violations of sparsity; a graph

³The paper [15] explains this geometric derivation, and its generalization to other groups, in more detail.

⁴This had been observed heuristically in [7], and is discussed in more detail in [13, Section 19.5].

is (k, ℓ) -spanning if it contains a spanning (k, ℓ) -graph; a (k, ℓ) -block is a subgraph that is a (k, ℓ) -graph; and a (k, ℓ) -basis is a maximal (k, ℓ) -sparse subgraph. As is standard, we refer to $(2, 3)$ -sparse graphs as *Laman graphs*.

Colored graphs are directed, so an edge ij means a directed edge from i to j . Their symmetric lifts are undirected, so the order of the vertices in an edge of the lift doesn't indicate orientation.

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2. Colored graphs and their lifts

In this short section, we quickly review some facts about colored graphs. Colored graphs are an efficient encoding of a (not-necessarily finite, undirected) graph $\tilde{G} = (\tilde{V}, \tilde{E})$ with a free Γ -action φ acting by automorphisms with finite quotient. We call the tuple (\tilde{G}, φ) a *symmetric graph*.

2.1. Symmetric graphs and colored quotients A straightforward specialization of covering space theory, which is given in detail in our paper [14, Section 9] with Justin Malestein, links colored and symmetric graphs: each colored graph *lifts* canonically to a *symmetric cover* \tilde{G} , and, after selecting representatives for the Γ -orbits of vertices, each symmetric graph (\tilde{G}, φ) determines a colored graph (G, γ) , with undirected graph underlying G being \tilde{G}/Γ .

2.2. The homomorphism ρ While the choice of colored quotient is not canonical, the rank of the image of the induced homomorphism $\rho : H_1(G, \mathbb{Z}) \rightarrow \Gamma$ is, which justifies the use of colored graphs in situations where the natural definition is in terms of symmetric graphs. To compute ρ on a cycle C , we traverse C in some direction, adding up the colors on the edges traversed forwards and subtracting the colors on edges traversed backwards. Since ρ is linear on $H_1(G, \mathbb{Z})$, it is determined by its images on any cycle basis of G . In particular, the fundamental cycles of any spanning forest F of G are a cycle basis, and so we can always assume that the colors are zero on F .

2.3. Edge orbits and colors If (\tilde{G}, φ) is a symmetric graph with colored quotient (G, γ) , we denote vertices the fiber over a vertex $i \in V(G)$ is given by \tilde{i}_γ , $\gamma \in \Gamma$, and the fiber over a (colored, oriented) edge $ij \in V(G)$ by $\tilde{i}_\gamma \tilde{j}_{\gamma + \gamma'}$, for $\gamma \in \Gamma$.

From the definition of the symmetric cover, we see that:

Lemma 2.1. *Let (G, γ) be a colored graph, and let (\tilde{G}, φ) be its lift. Then an edge $\tilde{i}_\gamma \tilde{j}_{\gamma'}$ is in (\tilde{G}, φ) if and only if either there is an (oriented) edge ij in G with color $\gamma' - \gamma$ or an oriented edge ji in G with color $\gamma - \gamma'$. \square*

2.4. Subgraph orbits We also require (cf. [1, Corollary 15]):

Lemma 2.2. *Let p be an odd prime, and let (G, γ) be a $\mathbb{Z}/p\mathbb{Z}$ -colored graph and let G' be a connected subgraph, and let \tilde{G}' be the lift of G' . Then \tilde{G}' is connected if and only if G' has non-zero ρ -rank.*

Proof. By selecting a spanning forest F of G that is also a spanning forest of G' , we may assume that G' is, in fact, all of G . If (G, γ) has ρ -rank zero, then, w.l.o.g., we may assume all the colors are zero, and the Proposition follows from the construction of the lift. Otherwise, we can pick a spanning tree T of G and some addition edge ij , so that the fundamental cycle C of ij in T has non-trivial ρ -image. The lift of C must be a collection of t cycles, with t dividing p , which is possible only if t is 1 or p ; the latter is only possible if ij is a self-loop with trivial ρ -image, contradicting how it was selected. Thus, the lift of (G, γ) contains p copies of T , connected by a cycle covering C . \square

A consequence is that if a and b have a common neighbor i , the element of Γ obtained by the oriented sum (in the sense of the definition of ρ) of the edge colors on the path from a to b via i can be “read off” from the neighbors of the vertices in the fiber over i in the lift \tilde{G} .

Lemma 2.3. *Let (G, γ) be a $\mathbb{Z}/p\mathbb{Z}$ colored graph, and let (\tilde{G}, φ) be its lift. Let i be a vertex with neighbors a and b in G , and let η be the sum of the colors along the path $a-i-b$ as defined for the map ρ . Then in the symmetric lift \tilde{G} , the neighbors $\tilde{a}_{\gamma'}$ and $\tilde{b}_{\gamma'}$ of any vertex \tilde{i}_{δ} in the fiber over i satisfy $\eta = \gamma' - \gamma$.*

Proof. Add the oriented edge ab to (G, γ) with color η to get a colored graph (H, γ) . The subgraph with the path from a to b via i and ab has, by construction, ρ -rank zero. Thus, Lemma 2.2 says that its lift is p vertex disjoint triangles. The lemma follows from applying Lemma 2.1 to the fiber over ab in the lift \tilde{H} . \square

3. The lift of a cone-Laman graph

This next proposition, which is a generalization of [1, Lemma 6], is our basic technical tool.

Proposition 1.1. *Let p be an odd prime, and let (G, γ) be a $\mathbb{Z}/p\mathbb{Z}$ -colored graph with n vertices and $2n - 1$ edges. Then (G, γ) is a cone-Laman graph if and only if its symmetric lift (\tilde{G}, φ) is Laman-sparse.*

Proof. We prove the contrapositive in both directions. First suppose that (G, γ) is not cone-Laman sparse. Minimal violations (i.e., cone-Laman-circuits) come in two types: Laman-circuits with trivial ρ -image and subgraphs with non-trivial image, n' vertices and $2n'$ edges. Lemma 2.2 says that the first type lifts to k copies of itself, blocking Laman-sparsity in the lift (\tilde{G}, φ) . The second type lifts to a subgraph of (\tilde{G}, φ) that has pn' vertices and $2pn'$ edges, which is certainly not Laman-sparse.

Now we suppose that the lift (\tilde{G}, φ) spans some Laman-circuit H . Denote by H_{γ} the image of H under $\varphi(\gamma)$, so that the orbit \tilde{H} of H is the union of the H_{γ} . If the H_{γ} are all disconnected from each other, then \tilde{H} is, by Lemma 2.2, the lift of a Laman-circuit with trivial ρ -image. Otherwise, again using Lemma 2.2, \tilde{H} is a graph on n' vertices made by gluing p Laman-circuits together in a ring-like fashion along Laman-sparse subgraphs. Thus, it has at most $p - 3$ Laman degrees of freedom and at least k Laman-dependent edges. In other words, a Laman-basis of \tilde{H} has at least $2n' - p$ edges and there are p other edges, implying that it has at least $2n'$ edges in total. \square

Our other technical tool is:

Proposition 3.1. *Let p be an odd prime, and let (\tilde{G}, φ) be a symmetric graph with a $\mathbb{Z}/p\mathbb{Z}$ -action φ . Suppose that H is a Laman-circuit in \tilde{G} , and suppose that, for some $\gamma' \in \mathbb{Z}/p\mathbb{Z}$, $\varphi(\gamma') \cdot H$ and H intersect on an edge ij . Then there is a Laman-circuit in \tilde{G} that goes through one edge in the orbit of ij .*

Proof. As in the proof of Proposition 1.1, denote by H_γ in the images of H under $\varphi(\gamma)$ and the whole orbit by \tilde{H} and adopt similar notation for ij . Because k is prime, γ' has order k , so we may, w.l.o.g., assume $\gamma = 1$. It follows that $H_\gamma \cap H_{\gamma+1}$ is never empty, so we may assume, w.l.o.g., that $(ij)_\gamma \in H_\gamma \cap H_{\gamma+1}$. Since Laman-circuits are Laman-spanning, and \tilde{H} is made by gluing Laman-circuits (the H_γ) along at least two vertices (the endpoints of $(ij)_\gamma$), it follows from [11, Theorem 5] that \tilde{H} is Laman-spanning as well.

The Proposition will follow from showing that \tilde{H} has a Laman-basis \tilde{L} that doesn't contain any edge in the orbit of ij , since the fundamental circuit of ij in \tilde{L} produces the desired circuit. We do this by refining the argument above. Let $L_1 = H - ij$. Since H is a Laman-circuit, L_1 is a Laman-graph, and it contains $(ij)_1$. Since H_1 is a Laman-circuit, $H'_1 = H_1 - (ij)_1$ is Laman-spanning, and thus, so is $L_1 \cup H'_1$. Because $(ij)_1$ is in the span of the Laman-block H'_1 , and, if ij is present in H_1 , it is in the span of the Laman-block L_1 , $L_1 \cup H'_1$ has a Laman-basis L_2 that does not contain ij or $(ij)_1$. Repeating this process p times, we obtain the desired \tilde{L} . \square

Remark The proof of Proposition 3.1 is written from the perspective of bases, but it can be argued directly from the perspective of circuits as well, obtaining a slightly different conclusion. We eliminate ij from the intersection of H and H_1 to obtain a circuit C_1 in \tilde{G} that does not go through ij but does contain $(ij)_1$. Iterating we obtain a family of circuits C_2, C_3, \dots, C_t such that C'_t does not contain $(ij)_\gamma$ for $\gamma < t'$. The process either ends at some $t < p$, yielding a circuit disjoint from the orbit of ij or at C_p , which contains only $(ij)_{k-1}$ from the orbit of ij .

4. Colored Henneberg moves

In this section we define the Henneberg moves that we will use, and the base graphs for each sparsity type.

4.1. The uncolored Henneberg moves If we forget about the colors, these are just the generalized Henneberg moves that can be found in [5, 11]; we will call these *uncolored Henneberg moves* **(H1)**, **(H1')**, and **(H2)** to distinguish them from the colored moves defined here. The following facts may be found in [5, 11]:

Proposition 4.1 ([5, 11]). *The uncolored Henneberg moves:*

- *Preserve $(2, 1)$ - and $(2, 2)$ -sparsity.*
- *Preserve Laman-sparsity when the neighbors of the new vertex are all distinct.*
- *Generate exactly $(2, 2)$ -graphs, starting from a doubled edge.*
- *Generate exactly $(2, 1)$ -graphs, starting from a vertex with a single self-loop.*

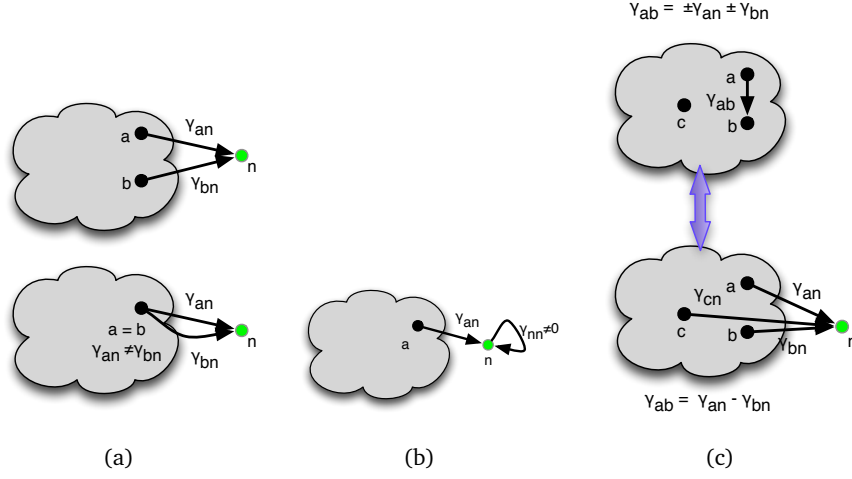


Figure 1: The colored Henneberg moves: (a) **(H1c)**; (b) **(H1c')**; (c) **(H2c)**.

4.2. Forward and reverse moves All the moves have forward and reverse directions. In each direction, we specify the allowed orientations and colors of any new edges. The forward moves can always be applied, while the reverse ones work only on a vertex of the appropriate degree.

4.3. The (H1c) and (H1c') moves We start with the simpler two moves. These involve adding one new vertex n and two new edges. For **(H1c)**, n is connected to the existing graph by two edges an and bn ; by convention we orient them into n , and, require that if $a = b$, the colors are different. The reverse move just removes a degree two vertex. The move **(H1c')** prime, which we give the suggestive mnemonic “lollipop move”, connects the new vertex n to the existing graph by one new edge an with arbitrary color and adds a self-loop on n with non-zero color. The reverse move simply removes a vertex incident on one self-loop and one other edge.

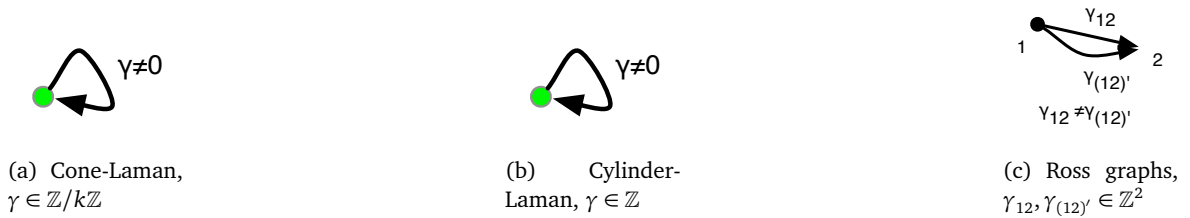


Figure 2: The base cases.

Remark The convention regarding the orientation in the forward direction doesn’t impose a restriction, since ρ -rank is preserved if we change the orientation of an edge and the sign of the color on the edge at the same time.

4.4. The (H2c) move The **(H2c)** move, which adds a new vertex n , removes one edge, and adds three new ones is slightly more complicated. Let ab be an edge with color γ_{ab} , and let c be

some other vertex. Note that a , b , and c , are not necessarily distinct. The forward **(H2c)** move removes the edge ab and replaces it with edges an and nb colored such that $\gamma_{an} - \gamma_{bn} = \gamma_{ab}$; an edge ac with arbitrary color is also added. If any of a , b , and c are the same, we further require that any parallel edges added have pairwise different colors.

The reverse direction is slightly more complicated. We don't have control over the orientation of the edges at the degree 3 vertex, and there are, potentially, several possibilities of the endpoints of the edge to put back, as well as a number of potential colors. We start with a degree-three vertex n , with neighbors a , b , and c , which, again, may not be distinct. A reverse **(H2c)** move removes n and adds back ab (resp. ac , bc) with some orientation and color that is the oriented sum, in the sense of the map ρ 's definition, of the oriented path short-circuited by ab (resp. ac , bc).

Remark In the proof that the reverse **(H2c)** move preserves cone-Laman sparsity, we will see that the color of the replacement edge is determined by the correspondence found in Lemma 2.1.

4.5. Base graphs We also have to specify the base cases of our induction. These are shown in Figure 2.

5. Theorem 1 for cone-Laman graphs

With the definition of the moves complete, we are in a position to prove Theorem 1 for cone-Laman graphs. This occupies the rest of the section. To set the notation, let p be an odd prime and let (G, γ) be a cone-Laman graph with $\mathbb{Z}/p\mathbb{Z}$ colors. The new vertex will be n .

5.1. Applicability of the colored Henneberg moves Because the colors come from $\mathbb{Z}/p\mathbb{Z}$, the ρ -rank of any subgraph is always zero or one. Since a cone-Laman graph has n vertices and $2n - 1$ edges, there is always a vertex of degree two or three. Thus, we need only to check that the moves defined in Section 4 preserve the cone-Laman property in the forward and reverse directions.

5.2. The base case It is readily seen that any of the claimed base cases is a cone-Laman graph. \square

5.3. Colored Henneberg moves and the symmetric lift We may interpret a colored Henneberg **(H1c)** or **(H2c)** move applied to G as a group of p uncolored Henneberg moves applied to \tilde{G} .

Lemma 5.1. *Let (H, γ) be the colored graph obtained from (G, γ) by applying an **(H1c)** move that adds a new vertex n and edges an and bn with colors γ_{an} and γ_{bn} . Then the symmetric lift \tilde{H} is obtained from the lift \tilde{G} by applying p **(H1)** moves.*

Proof. The degree of the vertices in the fiber over the new vertex n are all two, and the uncolored move **(H1)** adds a degree two vertex. \square

Lemma 5.2. *Let (H, γ) be the colored graph obtained from (G, γ) by applying an **(H2c)** move that adds a new vertex n , removes an edge ab with color γ_{ab} , and adds new edges an , bn , and cn with colors such that $\gamma_{an} - \gamma_{bn} = \gamma_{ab}$. Then the symmetric lift \tilde{H} is obtained from the lift \tilde{G} by applying p **(H2)** moves.*

Proof. The degree of the vertices in the fiber over the new vertex n are all three. Lemma 2.3 says that, for each $\gamma \in \mathbb{Z}/k\mathbb{Z}$ the neighbors of \tilde{n}_γ in \tilde{H} are determined by the colors γ_{an} , γ_{bn} , and γ_{cn} and that the neighbors of \tilde{n}_γ in the fibers over a and b are endpoints of an edge in the fiber over ab . This determines the data specifying an **(H2)** move for each edge in the fiber over ab . \square

5.4. The forward moves Now we check that the forward moves preserve the cone-Laman property. We will do this using the interpretation of the colored moves in terms of the lift \tilde{G} and uncolored moves and Proposition 1.1.

Lemma 5.3. *The **(H1c)** move, applied to (G, γ) , results in a cone-Laman graph.*

Proof. Let (H, γ) be the graph obtained after the move. The requirement that if the neighbors of the new vertex n are not distinct that the new edges have different colors says that, in the lift \tilde{H} , the neighbors of any vertex in the fiber over the new vertex n are distinct. Proposition 1.1, Lemma 5.1, and Proposition 4.1 imply that \tilde{H} is Laman-sparse. Since H has $2n - 1$ edges, the lemma follows. \square

The proof of the next Lemma is nearly identical, with Lemma 5.2 replacing Lemma 5.1, so we omit it.

Lemma 5.4. *The **(H2c)** move, applied to (G, γ) , results in a cone-Laman graph.* \square

The lollipop move **(H1c')** requires slightly more careful consideration of the lift \tilde{H} . There is no version of Lemma 5.2 for this move, because vertices in the fiber over the new vertex are neighbors with each other.

Lemma 5.5. *The **(H1c')** move, applied to (G, γ) , results in a cone-Laman graph.*

Proof. We consider the lift \tilde{H} . Any subset of t vertices in the fiber over the new vertex n spans at most t edges and connects to the rest of \tilde{H} with exactly t edges. Thus, for any $V' \subset V(\tilde{H})$ on n' vertices not in the fiber over n and t in the fiber over n , the number of edges induced by V' is bounded by $2n' - 3 + 2t = 2|V'| - 3$, since \tilde{G} is Laman-sparse by Proposition 1.1. \square

Remark The distinction between **(H2c)** and **(H1c')** is implicit in [20].

Remark With a slightly more delicate argument, using some structural results from [1, 15], we can show that the lemmas above hold even when p isn't prime by working with the colored graph directly. Since we don't need the extra generality, we omit the proof.

5.5. The reverse moves To complete the proof, we check that the reverse moves also preserve the cone-Laman property. In light of Proposition 1.1 and Lemma 5.1, the following are straightforward.

Lemma 5.6. *The reverse **(H1c)** move, applied to (G, γ) , results in a cone-Laman graph.* \square

Lemma 5.7. *The reverse **(H1c')** move, applied to (G, γ) , results in a cone-Laman graph.* \square

The hard step is the reverse **(H2c)** move, which only says that there is *some* edge we can put back with locally determined colors and orientation.

Lemma 5.8. *Given any degree-three vertex i in (G, γ) not incident on any self-loop, there is a reverse (H2c) move, applied to i , that results in a cone-Laman graph.*

Proof. Let (\tilde{G}, φ) be the symmetric lift, and let i be a degree three vertex in G with neighbors a , b , and c . Since we are doing a reverse (H2c) move (and not a lollipop), a , b , and c are all different from i (though not necessarily each other).

Let $\tilde{a}_\alpha, \tilde{b}_\beta, \tilde{c}_\gamma$ be the neighbors of \tilde{i}_0 . Proposition 1.1 tells us that these vertices are all different from each other, even if they are in a common orbit. Lemma 2.3 and Lemma 5.2 tell us that it is sufficient to show that if we can remove the fiber over i from \tilde{G} and add back the orbit of an edge between the neighbors of \tilde{i}_0 , the lemma will follow. Let \tilde{H} be the symmetric graph obtained by removing the fiber over \tilde{i}_0 from \tilde{G} .

Proposition 4.1 implies that there is an edge between some pair of $\tilde{a}_\alpha, \tilde{b}_\beta, \tilde{c}_\gamma$ that, when added to \tilde{H} , results in a Laman-sparse graph. Without loss of generality, this is $\tilde{a}_\alpha \tilde{b}_\beta$. The crux of the proof is that we can put back the entire orbit of $\tilde{a}_\alpha \tilde{b}_\beta$ maintaining Laman-sparsity. Let \tilde{H}' be the graph \tilde{H} with the orbit of $\tilde{a}_\alpha \tilde{b}_\beta$ added to it.

Suppose, for a contradiction, that \tilde{H}' is not Laman-sparse. Since $\tilde{H} + \tilde{a}_\alpha \tilde{b}_\beta$ is Laman-sparse, symmetry implies that any Laman-circuit in \tilde{H}' goes through two of the edges in the orbit of $\tilde{a}_\alpha \tilde{b}_\beta$. This is the situation from Proposition 3.1, leading to a contradiction: the new edges were selected so that there are no Laman-circuits through exactly one of them, but such a circuit is forced by Proposition 3.1. \square

6. Theorem 1 for cylinder-Laman graphs

We now turn to cylinder-Laman graphs. There are two differences, between this case and the cone-Laman one: we want the colors to come from \mathbb{Z} and we have to check that the two allowed moves can't generate a graph that is cone-Laman, but not cylinder-Laman. (It is clear that the lollipop move (H1c') does this.)

6.1. From $\mathbb{Z}/p\mathbb{Z}$ colors to \mathbb{Z} colors Instead of trying to replicate Proposition 1.1 on an infinite (\tilde{G}, φ) , we instead use the following reduction.

Lemma 6.1. *Let (G, γ) be a \mathbb{Z} -colored graph. Then (G, γ) is cone-Laman with \mathbb{Z} colors if and only if it is cone-Laman for $\mathbb{Z}/p\mathbb{Z}$ colors for some sufficiently large prime p .*

Proof. Pick the prime p large enough so that the magnitude of the colors arising in reverse steps is strictly less than p . \square

6.2. From cone-Laman to cylinder-Laman Cylinder-Laman graphs are characterized by [15, Theorem 8] as cone-Laman graphs that have a $(2, 2)$ -spanning underlying graph. This means the only thing to check is:

Lemma 6.2. *The (H2) move preserves the property of being $(2, 2)$ -spanning in the forward and reverse directions.*

Proof. The Tutte-Nash-Williams Theorem [17, 21] says that a $(2, 2)$ -spanning graph decomposes into two connected subgraphs. Since a degree three vertex will always be a leaf in exactly one of these subgraphs, for any such decomposition, the lemma is clear. \square

7. Theorem 1 for Ross-graphs

Finally, we adapt our technique to Ross graphs, recovering a result of [19].

7.1. The base case Checking that the claimed bases are Ross graphs is straightforward. \square

7.2. Inductive step Since the underlying graphs of Ross graphs are $(2, 2)$ -graphs [1, Lemma 4], by the Henneberg construction for $(2, 2)$ -graphs [5, 11], we just need to check the **(H1c)** and **(H2c)** moves in each direction. The proof for **(H1c)** is identical to that in Section 5, as is the forward direction for **(H2c)**.

For the reverse direction, we can't directly apply Proposition 1.1, unless the colored graph (G, γ) has ρ -rank one. However, if it does not, we can make the following modification.

Proposition 7.1. *Let p and q be distinct odd primes, and let (G, γ) be a $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ -colored graph with $2n - 1$ edges. Then (G, γ) is cone-Laman if and only if its lift is Laman-sparse.*

Proof. Lemma 2.2 has the following refinement: the number of connected components in the lift of (G, γ) is the index of its ρ -image in Γ [13, Lemmas 5.4 and 5.5]. In this case, the only possibilities for the index are p , q , and 1. The rest of the proof of Proposition 1.1 then goes through. \square

The reductions from Ross-graphs to cone-Laman graphs then goes through using the steps from Section 6.

8. Conclusions

We conclude we several questions and potential directions.

8.1. Henneberg constructions for all cone-Laman graphs For cylinder-Laman graphs Theorem 1 settles the question of inductive constructions. We also gave a new, pleasant, proof of an existing characterization for Ross graphs. We note, however, that while cone-Laman graphs have rigidity characterizations for colors in $\mathbb{Z}/k\mathbb{Z}$ for any $k \geq 2$, Theorem 1 only applies to $\mathbb{Z}/p\mathbb{Z}$.

Question 1. *Give a Henneberg construction for cone-Laman graphs with colors from any group $\mathbb{Z}/k\mathbb{Z}$.*

The main difficulty seems to be when the ρ -image of a subgraph has order two. This causes Proposition 1.1 to fail, so it will require a different argument.

8.2. Ross-circuits and global rigidity The most natural application of these Henneberg moves on colored graphs would be in characterizing *global rigidity* (see e.g., [3, 6]) for fixed-lattice frameworks [13, Section 19.1], [19]. We are unaware of a detailed conjecture for the right class of graphs. However, Bruce Hendrickson's proof that a globally rigid planar framework must be redundantly rigid [8] extends to the fixed-lattice setting. This tells us that Ross-circuits will play a role.

Question 2. *Give an inductive characterization of Ross-circuits.*

It seems, by analogy with [2], plausible that **(H2c)** and 2-sum are sufficient.

8.3. Generalizing Proposition 1.1 A more combinatorial direction relates to generalizing the relationship that holds between the cone-Laman and Laman matroids. Recalling (4), we see that this generalizes the bounding function “ $kn' - \ell$ ” from (k, ℓ) -sparsity in two ways: the dimension of the ρ -image determines a positive adjustment; and the ρ -image of each connected component determines a negative adjustment. We are unaware of families like this having been studied before, though [22, “Matroid Theorem”] appears to contain part of the story.

In fact, [14] extends this idea further, allowing non-abelian groups Γ that admit a kind of “uniform matroid” structure [14, Section 8]. Thus, since the colored graph families treated here are part of a much more general phenomenon, we ask:

Proposition 8.1. *Can one characterize sparsity matroids on colored graphs in terms of an appropriate “matroid lift” that does not explicitly reference the colors on the base graph?*

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